Normalized harmonic map flow

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Harmonic maps

Let (M, g), $(N, h) \hookrightarrow \mathbb{R}^n$ be closed (compact, without boundary) Riemannian manifolds, with smooth nearest-neighbor projection $\pi_N \colon U_\delta N \subset \mathbb{R}^n \to N$ for some $n \in \mathbb{N}$ and $\delta > 0$.
For a smooth map $u \colon M \to N$ the Dirichlet energy is given by

$$D(u) = \frac{1}{2} \int_M |\nabla_g u|^2 d\mu_g .$$

The map u is harmonic if u is a critical point of D in the sense that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} D(\pi_N(u + \varepsilon\varphi)) = -\int_M \Delta_g u \cdot d\pi_N(u)\varphi\mu_g = 0$$

for any variation $\varphi \in C^1(S^3; \mathbb{R}^n)$; that is, if the tension field

$$\tau(u) := d\pi_N(u)\Delta_g u = \Delta_g u + A(u)(\nabla u, \nabla u) = 0,$$

with $A \colon T_pN \times T_pN \to T_p^\perp N$ the 2^{nd} fundamental form of N.

Heat flow

Eells-Sampson (1965): If N is non-positively curved, for any given non-positively curved, for any given smooth map $u_0 \colon M \to N$ there is a harmonic map

$$u_\infty = \lim_{t \to \infty} u(t) \colon M \to N$$

homotopic to u_0, where $u(t)$ solves the harmonic map heat flow

$$u_t = \tau(u)$$

$$u_t = \tau(u) \quad \Big(= \Delta_g u + A(u)(\nabla u, \nabla u) \in T_u N \Big)$$

with initial data u_0, so that

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} D(u(t)) &= - \int_M \Delta_g u \cdot u_t \; d\mu_{g_M} \\ &= - \int_M \tau(u) u_t \; d\mu_g = - \int_M |u_t|^2 \; d\mu_{g_M}. \end{aligned}$$

Heat flow to arbitrary closed targets

M.S. (1985): If $dim(M) = 2$, for any smooth $u_0 \colon M \to N$ there is a unique, global, partially regular weak solution to the harmonic map heat flow $u \colon \mathbb{R}_+ \times M \to N$ with $u(0) = u_0$ and non-increasing energy. The at most finitely many singular points in space-time of the flow are characterized by the "bubbling-off" of non-constant smooth harmonic maps $S^2 \to N$, arising as limits from the suitably rescaled flow. "Bubbling": Chang-Ding-Ye (1992), Topping (2002).

M.S. (1988) for $dim(M) \geq 3$ proved monotonicity formula and ε-regularity for harmonic map heat flow.

Chen-Struwe (1989) with the help of the results from M.S. (1988), showed existence of a global, partially regular weak solution u of the harmonic map heat flow which is smooth away from a closed singular set of finite $(m-2)$-dimensional Hausdorff measure.

See also the book by Changyou Wang and Fanghua Lin (2008).

Harmonic spheres, $dim(M) \geq 3$

For $\bar{u} = id\colon S^m \to S^m$, $m \geq 3$, Eells-Sampson observed that

$$\inf_{\gamma \in \Gamma} D(\bar{u} \circ \gamma) = 0,$$

where Γ is the group of Möbius transformations of S^m.

Chen-Ding (1990) used this fact together with monotonicity and ε-regularity to produce examples of (topologically non-trivial) smooth maps $u_0\colon S^m \to S^m$ for which the harmonic map heat flow blows up in finite time T. (In fact, $T \leq CD(u_0)^{2/(m-2)}$ (M.S.)).)

El Soufi (1995): If $u\colon S^m \to N$, $m \geq 3$, is smooth, non-constant, and harmonic, we have

$$D(u) = \sup_{\gamma \in \Gamma} D(u \circ \gamma),$$

and equality holds if and only if $\gamma = id$ (up to rotations).

Hence, for any N, any smooth, non-constant harmonic map $u\colon S^m \to N$ is unstable under the heat flow when $m \geq 3$.

Normalized harmonic map flow

For (smooth) $u\colon S^m \to N$ let

$$E(u) = \sup_{\gamma\in\Gamma} D(u\circ\gamma).$$

Note: E detects topology; e.g. $\inf_{u\sim id\colon S^m\to S^m} E(u) > 0$.

Aim: For initial data $u_0 \in C^{2+\alpha}(S^3; N)$ with $D(u_0) = E(u_0)$ we seek a to evolve u_0 through a flow $u = u(t)$, $t \geq 0$, with $u(0) = u_0$ such that

$$D(u(t)) = E(u(t)) \text{ for all } t \geq 0, \tag{1}$$

and such that $D(u(t))$ is strictly decreasing in time t.

Key idea: Quite surprisingly, under certain conditions it is possible to characterize (1) as a balancing condition for the center of mass, which can be maintained by modifying the heat flow via an accompanying flow in the Möbius group Γ.

We now explain this in the case when $m = 3$.

Stereographic coordinates

Fix $m = 3$. Define stereographic projection $\pi\colon S^3 \subset \mathbb{R}^4 \to \mathbb{R}^3$ by

$$\pi(x) = \frac{x'}{1 + x^4},\ x = (x', x^4) \in S^3 \subset \mathbb{R}^4,$$

with inverse $\psi\colon \mathbb{R}^3 \to S^3$ given by

$$\psi(x) = \frac{(2x, 1 - |x|^2)}{1 + |x|^2},\ x \in \mathbb{R}^3.$$

Computing

$$\psi^* g_{S^3} = \Big(\frac{2}{1 + |x|^2}\Big)^2 g_{\mathbb{R}^3},$$

for any $u\colon S^3 \to N$ with $\hat{u} = u \circ \psi\colon \mathbb{R}^3 \to N$ we then have

$$\begin{aligned} D(u) &= \frac{1}{2} \int_{S^3} |\nabla_{g_{S^3}} u|^2 d\mu_{g_{S^3}} \\ &= \frac{1}{2} \int_{\mathbb{R}^3} |\nabla_{\psi^* g_{S^3}} \hat{u}|^2 d\mu_{\Psi^* g_{S^3}} = \int_{\mathbb{R}^3} \frac{1}{1 + |x|^2} |\nabla \hat{u}|^2 dx. \end{aligned}$$

Scaling

Fixing the axis $a = e_4$, for any $s > 0$, letting

$$\delta_s(x) = s^{-1}x,\ x \in \mathbb{R}^3,$$

and setting

$$\gamma_s = \psi \circ \delta_s \circ \pi \colon S^3 \to S^3,$$

we introduce the scaled map

$$u_s = u \circ \gamma_s = u \circ \psi \circ \delta_s \circ \pi \colon S^3 \to N,$$

represented by

$$\hat{u}_s = u_s \circ \psi = \hat{u} \circ \delta_s \colon \mathbb{R}^3 \to N,$$

with

$$\begin{aligned} D(u_s) &= \int_{\mathbb{R}^3} \frac{1}{1+|x|^2} |\nabla(\hat{u} \circ \delta_s)|^2 dx \\ &= \int_{\mathbb{R}^3} \frac{s^{-2}}{1+|x|^2} (|\nabla \hat{u}|^2 \circ \delta_s) dx = \int_{\mathbb{R}^3} \frac{s}{1+s^2|x|^2} |\nabla \hat{u}|^2 dx, \end{aligned}$$

and similarly for any axis $a \in S^3$.

Center of mass

Recall

$$D(u_s) = \int_{\mathbb{R}^3} \frac{s}{1+s^2|x|^2} |\nabla \hat{u}|^2 dx.$$

Let

$$\beta(s) = \frac{s}{1+s^2|x|^2} \quad \text{with} \quad \partial_s\big|_{s=1}\beta(s) = \frac{1-|x|^2}{(1+|x|^2)^2}.$$

Also define

$$\begin{aligned} X(u) &= \frac{1}{2}\int_{S^3} x|\nabla_{g_{S^3}} u|^2 d\mu_{g_{S^3}} \\ &= \frac{1}{2}\int_{\mathbb{R}^3} \psi(x)|\nabla_{\psi^* g_{S^3}} \hat{u}|^2 d\mu_{\psi^* g_{S^3}} = \int_{\mathbb{R}^3} \frac{(2x, 1-|x|^2)}{(1+|x|^2)^2} |\nabla \hat{u}|^2 dx, \end{aligned}$$

the center of mass (up to the scale factor $D(u)$). Then we have

$$\partial_s\big|_{s=1} D(u_s) = X(u)\cdot e_4 = \int_{S^3} \tau(u) du \cdot \partial_s\big|_{s=1}\gamma_s \, d\mu_{g_{S^3}}. \tag{2}$$

By (2) then, for u to be harmonic, necessarily $X(u) = 0$.

First variation in s

For a fixed $u \in C^2(S^3; N)$ set $f(s) = D(u_s)$ with $f' = df/ds$. Suppose that $D(u) = E(u) = \sup_{a \in S^3,\, 0<s<\infty} D(u \circ \gamma_{a,s})$ so that

$$0 = f'(1) = \partial_s\big|_{s=1} D(u_s) = \int_{\mathbb{R}^3} \tau_{\psi^* g_{S^3}}(\hat{u}) d\hat{u} \cdot x \; d\mu_{\psi^* g_{S^3}},$$

where $\tau_{\psi^* g_{S^3}}(\hat{u}) = \tau(u) \circ \psi$. Then

$$\begin{aligned} f'(s) = &\int_{\mathbb{R}^3} \frac{(1-s^2)|x|^2}{1+s^2|x|^2} \tau_{\Psi^* g_{S^3}}(\hat{u})(d\hat{u} \cdot x) d\mu_{\Psi^* g_{S^3}} \\ &+ \int_{\mathbb{R}^3} \frac{(1-s^2)(1+|x|^2)^2}{2(1+s^2|x|^2)^2} |d\hat{u} \cdot x|^2 d\mu_{\Psi^* g_{S^3}}, \end{aligned} \tag{3}$$

and hence (for $0 < s < 1$)

$$\frac{f'(s)}{1-s^2} \geq \frac{1}{4} \int_{\mathbb{R}^3} \left(|d\hat{u} \cdot x|^2 - 4|\tau_{\Psi^* g_{S^3}}(\hat{u})|^2 \right) d\mu_{\Psi^* g_{S^3}}. \tag{4}$$

Two important conclusions

From (3), (4) we immediately deduce El Soufi's result: If u is smooth, non-constant, and harmonic, we have

$$\forall a \in S^3,\ s > 0\colon D(u) \geq D(u \circ \gamma_{a,s}),$$

with strict inequality unless $s = 1$.

Moreover, if u with $X(u) = 0$ is δ_0-uniformly 3-dimensional in the sense that

$$\forall a \in S^3\colon \|du \cdot \frac{d}{ds}\big|_{s=1}\gamma_{a,s}\|_{L^2} = \|d\hat{u} \cdot x\|_{L^2} \geq \delta_0,$$

and if u satisfies $2\|\tau(u)\|_{L^2} < \delta_0$, then

$$(1-s)f'(s) > 0 \text{ for } s \neq 1$$

and we conclude

$$D(u) = E(u) = \sup_{\gamma \in \Gamma} D(u \circ \gamma). \tag{5}$$

Normalized harmonic map flow

For initial data $u_0 \in C^{2+\alpha}(S^3; N)$ with $X(u_0) = 0$ we seek a solution $u = u(t)$ with $u(0) = u_0$ such that

$$u_t = \tau(u) + \xi \cdot du, \tag{6}$$

where

$$\xi = \Xi(\sigma) := \sum_{i=1}^{4} \sigma_i \frac{d}{ds}\Big|_{s=1} \gamma_{e_i,s} = \sum_{i=1}^{4} \sigma_i R_i^{-1} \frac{d}{ds}\Big|_{s=1} \gamma_s \circ R_i$$

for $R_i \in SO(4)$ chosen so that with the standard orthonormal basis vectors e_i for $\mathbb{R}^4$ we have $R_i e_i = e_4$, $1 \le i \le 4$, and where the numbers σ_i, $1 \le i \le 4$, are determined so that

$$\frac{dX(u)}{dt} = 0 \tag{7}$$

and hence

$$X(u) \equiv 0. \tag{8}$$

A remark on normalization

In view of $X(u)\cdot\rho = dD(u)(\eta\cdot du)$ for any $\rho\in\mathbb{R}^4$ with $\eta=\Xi(\rho)$, condition (7) requires that $\xi=\Xi(\sigma)$ satisfies the equation

$$0 = \frac{d}{dt}(X(u)\cdot\rho) = d^2D(u)(u_t, \eta\cdot du) + \int_M \tau(u)\,\eta\cdot du_t \; d\mu_{g_{S^3}}$$
$$= d^2D(u)(\tau(u)+\xi\cdot du, \eta\cdot du) + \int_M \tau(u)\,\eta\cdot d(\tau(u)+\xi\cdot du)\; d\mu_{g_{S^3}}$$

for any $\rho\in\mathbb{R}^4$; that is,

$$d^2D(u)(\xi\cdot du, \eta\cdot du) + \int_M \tau(u)\,\eta\cdot d(\xi\cdot du)\; d\mu_{g_{S^3}}$$
$$= -d^2D(u)(\tau(u), \eta\cdot du) - \int_M \tau(u)\,\eta\cdot d\tau(u)\; d\mu_{g_{S^3}}.$$

Local existence in a perturbative regime

Definition

A δ_0-uniformly 3-dimensional map u has *Möbius non-degenerate* second variation $d^2D(u)$, if the quadratic form

$$Q_u(\sigma, \rho) = d^2D(u)(\xi \cdot du, \eta \cdot du),$$

with $\xi = \Xi(\sigma)$, $\eta = \Xi(\rho)$ for $\sigma, \rho \in \mathbb{R}^4$, is non-degenerate.

Theorem

Assume $u_0 \in C^{2+\alpha}(S^3; N)$ with $X(u_0) = 0$ is δ_0-uniformly 3-dimensional for some $\delta_0 > 0$ and has Möbius non-degenerate second variation $d^2D(u_0)$ with $(2 + C_0)\|\tau(u_0)\|_{L^2} < \delta_0$ for some $C_0 > 0$ depending on $d^2D(u_0)$. Then there is $T > 0$ and a unique smooth solution $u = u(t) \in C^{2+\alpha,1+\alpha/2}(S^3 \times [0, T]; N)$ of (6), (7) for suitable $\sigma = (\sigma_i(t))$ with $u(0) = u_0$, and $D(u(t)) = E(u(t))$ is non-increasing in time.

Stability of the identity map

Theorem

For any smooth initial data $u_0 \in C^{2+\alpha}(S^3; S^3)$ *with* $X(u_0) = 0$ *and sufficiently close to* $\bar{u} = id$ *there exists a global, smooth solution* u *to* (6), (7) *with suitable* $\xi = \Xi(\sigma(t))$*, and normalized with respect to rotations; moreover,* $u(t) \to \bar{u}$ *smoothly as* $t \to \infty$.

Note: By Smith (1975), infinitesimal rotations and variations in the Möbius group are responsible for nullity and index of $d^2D(\bar{u})$.

This result is stable with respect to perturbations of the geometry.

Theorem

There is $\nu > 0$ *such that if* $N \subset \mathbb{R}^4$ *admits a diffeomorphism* $\phi_N \colon N \to S^3$ *with* $\|\phi - id\|_{C^{2+\alpha}} < \nu$*, and if* $u_0 \in C^{2+\alpha}(S^3; N)$ *with* $X(u_0) = 0$ *satisfies* $\|\phi_N \circ u_0 - id\|_{C^{2+\alpha}} < \nu$*, there is a unique, global, smooth solution* $u = u(t)$ *of* (6), (7) *for suitable* $\xi = \Xi(\sigma(t))$ *with* $u(0) = u_0$*, and a smooth harmonic map* $u_\infty \colon S^3 \to N$ *such that* $u(t)$ *smoothly converges to* u_∞ *as* $t \to \infty$.

Stability of the Hopf map

Consider $S^3 = \{(w,z) \in \mathbb{C}^2 \cong \mathbb{R}^4;\ |w|^2 + |z|^2 = 1\}$, and let $u_H \colon S^3 \to S^2 = \{(p,r) \in \mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3,\ |p|^2 + |r|^2 = 1\}$ be the Hopf map given by

$$u_H(w,z) = (2w\bar{z}, |w|^2 - |z|^2).$$

Theorem (w.i.p. with Tobias Lamm)

For any smooth initial data $u_0 \in C^{2+\alpha}(S^3; S^2)$ *with* $X(u_0) = 0$ *and sufficiently close to* u_H *there exists a global, smooth solution* u *to* (6), (7) *with suitable* $\xi = \Xi(\sigma(t))$, *normalized with respect to rotations in the domain and in the image, such that* $u(t) \to u_H$ *smoothly as* $t \to \infty$.

Remark: Again we use Smith's (1975) characterization of the index and nullity of $d^2D(u_H)$.

An example for blow-up

For $b > 0$ let $N(b)$ be the ellipsoid

$$N(b) = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R};\ b^2|x|^2 + |y|^2 = b^2\}.$$

Then by a further result of Smith (1975) for sufficiently large $b > 1$ there is no co-rotational harmonic map $u\colon S^3 \to N(b)$, symmetric with respect to the equator.

For any co-rotational, symmetric map u there holds $X(u) = 0$.

Thus, for $u_0 = \Sigma id\colon S^3 \to N(b)$, the "suspension" of the identity, as defined by Smith, the normalized harmonic map flow (like the standard flow) has to blow up in finite or infinite time.

Global theory, non-smooth flows

Open questions: i) If $u_0 \in C^2(S^3; N)$ with "large" tension field $\tau(u_0)$, or if u_0 only is of class H^1, we cannot hope to achieve $D(u(t)) = E(u(t))$ or $X(u) = 0$ in a smooth fashion for all time. Is there a natural definition of weak normalized harmonic map flow that also allows for discontinuous changes of u to $u \circ \gamma$ (in addition to singularities as in the harmonic map heat flow)?

ii) Can one show that (6), (7) is well-posed for initial data of class H^1 with $X(u_0) = 0$ and $E(u_0) < \varepsilon_0 \ll 1$? How big is ε_0?

iii) What happens for large data? Can one use Zhu's (2018) "moving-centre monotonicity formulae" to characterize possible singularities?

Reference: M. Struwe: *Normalized harmonic map heat flow*, Comm. Pure Appl. Math. 73 (2020), no. 3, 664-686.